Beauty in mathematics

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INTRODUCTION

Mathematicians often refer to certain pieces of mathematics using words like beautiful, elegant, cumbersome, or ugly. Such references appear in discussions, in teaching contexts and in the literature, especially the non-technical literature. However, serious, more rigorous investigation of 'mathematical beauty' is much less common; it even seems that mathematicians are mostly off-duty when they speak of mathematical beauty. Here I intend to adopt a 'serious' approach: the aim of this work is to develop an aesthetic theory that accounts for aesthetic judgements in mathematics. In this work I do not attempt to underline the importance of mathematical beauty or to show the reader instances of it[1]. Nor do I attempt to justify the fact that the use of terms like 'beautiful' or 'elegant' coincides with the general precepts of the mathematical discipline. I believe that when mathematicians talk about mathematical beauty they are still being 'serious' mathematicians. Assuming this is no simple matter, however.

When mathematicians employ the word 'beauty' to qualify a theory, for example, people unacquainted with mathematics can surely find this usage strange. Beauty in mathematics seems to differ from the beauty we encounter in our everyday life. In these circumstances, we are perfectly justified to ask: what do mathematicians mean when they talk about beauty? I intend to answer this question. I intend to defend the position that judgements of beauty in mathematics are genuine aesthetic judgements. How I intend to defend this position is more complicated: I shall develop a theoretical framework that accounts for mathematical aesthetics in the same fashion as it account for 'regular' aesthetics. My answer to the question above is thus almost trivial: what mathematicians do when they talk about beauty is to express genuine aesthetic reactions. To ground this answer I interpret aesthetic terms as follows: the term 'beautiful' (and other aesthetic terms), in addition to referring to the property of *being beautiful*, has the function of

[1] In Chapter 4 I analyze three examples of mathematical beauty. However, the purpose of that analysis is to show how the theory proposed in Chapter 3 can be applied, not to show the beauty of those items. There is a large body of literature that accomplishes the display of mathematical beauty much better than this work. A few examples are: Nahin (1998), Nahin (2006), Le Lionnais (2004), Wells (1990).

elucidating to ourselves and to our interlocutors the subjective state caused in us by the object qualified as beautiful[2].

Hence, I shall defend the more specific claim that aesthetic judgements are characterized by two things: their conditions of application and their role in our experiencing objects and properties. The conditions of application depend on the person's subjective state. The role, or rather the roles, that aesthetic judgements play are to elucidate subjective experience and to communicate the results of this elucidation. For example, when a mathematician qualifies a proof as beautiful, his description 'this proof is beautiful' conveys the information that the proof possesses the property of being beautiful. But more importantly, the uttering of this description helps the mathematician to clarify and identify his subjective state as an *affective state* similar to the states we experience in normal occurrences of beauty. Chapter 3 is devoted to an explication of all this in detail.

The broad aim of this thesis is thus to provide an interpretation of aesthetic terms (words like beautiful, elegant, ugly, etc.) and judgements which accounts for their use in mathematics in the same manner as it accounts for their 'normal' use: the aim is to justify the literal interpretation of mathematical aesthetic judgements.

My central proposal is that the key to this interpretation is to approach the subject of aesthetics from a systemic perspective: from this perspective, things like aesthetic pleasure, aesthetic experience or aesthetic judgements, are elements that interact with each other, forming a larger system. This system is an *aesthetic-process* and its development involves objects, subjects, social interaction and historical and cultural influences. The idea of aesthetic-process serves to flesh out an aesthetic theory.

Now, non-literal approaches to aesthetic terms in mathematics, approaches that re-interpret the term 'mathematical beauty', can be found in the literature, and they illuminate important aspects of mathematical beauty. Metaphorical uses of aesthetic terms exist. Sometimes mathematicians (or scientists in general) intend to emphasize the heuristic or practical importance of certain pieces of mathematics by using references to beauty. In these cases the word beauty is a metaphor that refers to a general kind of virtue, not necessarily an aesthetic quality. Richard Feynman, for example, has referred to

[2] I call this function of aesthetic terms *articulation*.

Euler's formula[3] as "the most remarkable formula of mathematics – our jewel" (Feynman, Leighton and Sands, 1963, p. 23). Although an aesthetic interpretation of this statement is possible, I believe that a different, non aesthetic, interpretation of it is equally plausible. Euler's formula has enormous repercussions in fields like engineering or applied physics. The word 'jewel' can refer, so to speak, to the 'economic' value of a jewel. We can contrast this with a more obvious and explicitly literal use of aesthetic terms by Bertrand Russell: "Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture" (Russell, 2004, p. 47).

The distinction between literal and non-literal approaches is more evident when seen from our 21st century point of view. Earlier historical periods, however, did not distinguish so sharply between what we nowadays know as scientific and artistic disciplines. It is well known that for Pythagoreans the harmonic nature of the world was clearly manifested in numbers as well as in music. Without a distinction between harmony in numbers and music, distinguishing between literal or non-literal interpretations of the term 'harmony' makes no sense; the same can be said, I believe, of the term 'beauty'. During the middle ages, scientific education also incorporated this relationship: music, along with arithmetic, geometry and astronomy formed the quadrivium: the group of scientific subjects taught in mediaeval universities, after the literary education of the trivium which consisted of grammar, dialectic (logic) and rhetoric. The inclusion of music among the scientific disciplines was not arbitrary; it was grounded on a conception of knowledge. The Greek philosopher Proclus Lycaeus (412-485) dealt with the subjects of the quadrivium by explicating them based on the notion of quantity; for Proclus the quadrivium's subjects were mathematical subjects (Gow, 2004). By dissociating discrete from continuous quantity, Proclus believed that an arithmetical fact had its analogue in geometry and vice versa; and that a musical fact had its analogue in astronomy and vice versa. The subjects were characterized as follows: Arithmetic is

[3] In technical works the expression $e^{i\pi}=cos\ x + i\ sin\ x$ is referred to as Euler's formula and the special case $e^{i\pi}=-1$, or equivalently $e^{i\pi}+1=0$, is referred to as Euler's identity (Weisstein 2003, p. 952). However, in their papers on mathematical beauty David Wells and Francois Le Lionnais call $e^{i\pi}=-1$ Euler's formula. Since Wells and Le Lionais are frequently cited here, providing examples of mathematical beauty, I follow their convention and I refer to $e^{i\pi}=-1$ as *Euler's Formula*. I shall reiterate this clarification when needed.

discrete quantity at rest. Geometry is continuous quantity at rest. Music is continuous quantity in motion. Astronomy is discrete quantity in motion (Gow, 2004, pp. 71-72).

The 6th century philosopher Anicius Manlius Severinus Boethius (ca. 480–524 or 525) was the one who introduced the word ‘quadrivium’, which means four-fold path (Marenbon, 2003, p.14). He translated Euclid and Ptolemy’s texts that were used in the teaching of the quadrivium in mediaeval universities. Boethius wrote his own treatise on music, the “Principles of Music”. He distinguished three types of music: instrumental, human and cosmic, all of which involved the study of harmonic ratios. Boethius was not concerned with the practice of music but with its principles, something roughly similar to what we call music theory nowadays, except that these principles embodied “ideal structures of the world” (Marenbon, 2003, p.15). Boethius believed that human beings find a natural joy in music, and he linked this fact to Plato’s view that the world is structured according to musical intervals. He also endorsed Plato’s view of the power that music has to change people’s moods and behaviour. As to Proclus, music to Boethius is a mathematical subject, and yet music is characterized by the same features we attribute to it today: the power to deliver enjoyment and affect our emotions. To Boethius there seems to be no conflict, but rather a natural link between beauty and mathematics. There was no need for a re-interpretation of mathematical beauty.

The gap between music and mathematics deepened as the systems of arts and sciences began to develop into the two contrasting fields we know today. The Renaissance and Modernity granted less importance to the aesthetic aspects of mathematics. At the same time, aesthetic problems began to be seen as independent of cognitive activities; the birth of modern aesthetics is characterized by the introduction of the view that the phenomena related to our perceptions of beauty are independent of any practical or cognitive concerns (Guyer, 2004). As the empirical component of knowledge gained importance in science and its power of description and prediction was notably enhanced by mathematics, mathematics became more closely linked to the other sciences. Mathematics’ old relationship with music lost relevance. By the end of the 19th century the epistemic problems were at the centre of the philosophical debate on mathematics (Colyvan, 2003).

The relation between mathematics and beauty would not be investigated for some time. The peculiarities of mathematics, a formal discipline with no relation to empirical events and governed solely by logic, posed the most serious difficulties for philosophers at the end of the 19th century. Gottlob Frege, for example, considered that arithmetic and games of chess were very alike:

> An arithmetic with no thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously because it expresses no thought. Why can arithmetic equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now, it is the applicability alone which elevates arithmetic from a game to the rank of science. So applicability necessarily belongs to it (Frege, 1903, p. 291).

A purely formalist approach to mathematics does not allow us to justify the place of mathematics among the sciences. This is a worrying picture for anyone who regards mathematics as a serious discipline. Frege concluded that it is applicability which elevates arithmetic from a game to a science.

Mathematical beauty seems a frivolous concern when urgent issues like the scientific status of mathematics are at hand. If mathematical beauty is to be addressed, a serious mathematician should address it in such a way that it does not compromise the scientific character of mathematics; this concern, I believe, would explain why contemporary mathematicians might be interested in interpreting mathematical beauty in terms of scientific precepts. Gian-Carlo Rota (Rota, 1997), offers a notorious example of a scientific-character-preserving approach to mathematical beauty. Rota believes that when a mathematician employs the term ‘mathematical beauty’, he is actually referring to the objective property of being enlightening, of giving us a deeper understanding: mathematical beauty is still a ‘serious’ business.

However, another trend gained appeal during the 20th century: the divide between scientific and aesthetic concerns began to be questioned; we witnessed attempts to make reason and beauty meet again. Nelson Goodman (Goodman, 1968), for example, proposed that cognitive processes play an important role in aesthetic appreciation. Susan Langer (Langer, 1942) addressed music by analyzing topics like language, abstraction, and knowledge. Some authors in the philosophy of science have also shown an interest in the topic of beauty in science. James McAllister (McAllister, 1996) developed a rationalist picture of scientific change based on the evolution of aesthetic canons through a mechanism that he called aesthetic induction. Theo Kuipers (Kuipers, 2002), when further developing the idea of aesthetic induction, explored the idea that beauty can play a role in the scientific search for truth.

A rapprochement between aesthetics and mathematics can also be found among mathematicians. François Le Lionnais (Le Lionnais, 1948), for example, resorts to the history of art to illustrate different kinds of mathematical beauty. He considers that beauty appears in every branch of knowledge but nowhere with more force than in mathematics. Le Lionnais endorses a literal approach to mathematical beauty: "Has not the Western world confirmed the opinion of ancient Greece which up to the time of Euclid considered mathematics more an art than a science?" (Le Lionnais, 2004, p. 121). He presents an interesting list of authors who enthusiastically expressed their aesthetic views on mathematics. But, even more importantly, Le Lionnais recognizes the need to deal with the subject in a more rigorous manner, not just by remarking and showing the existence of mathematical beauty:

> If some great mathematicians have known how to give lyrical expression to their enthusiasm for the beauty of their science, nobody has suggested examining it as if it were the object of an art – mathematical art– and consequently the subject of a theory of aesthetics, the aesthetics of mathematics (Le Lionnais, 2004, p. 122).

Le Lionnais emphasizes that there is much work to do to accomplish an aesthetics of mathematics; his own study "has no intention of establishing [an aesthetics of mathematics]; it aspires only to prepare the way for it" (Le Lionnais, 2004, p. 122). Le Lionnais is not only a supporter of a literal interpretation of mathematical beauty; he also favours the development of a more rigorous approach to an aesthetic theory, not just resigning ourselves to pre-theoretical displays of enthusiasm. If we take Le Lionnais seriously, then an aesthetics of mathematics must not only address mathematical beauty in a literal way, but also in a fully theoretical way.

Now, pre-theoretical approaches cannot be considered as accounts of mathematical beauty, but they can illuminate aspects of it. For example, Russell's analogy comparing mathematical beauty to sculpture gives us the rough idea that mathematical beauty is contemplative, related, perhaps, to the static character of abstract mathematical objects. Many pre-theoretical approaches are just brief remarks accompanying non-technical works. Paul J. Nahin (Nahin, 2006), for example, comments very briefly that he considers mathematical beauty as related to the fact that mathematics is *disciplined* reasoning:

> The reason I think that Einstein's theory is still beautiful (despite currently being replaced by quantum-mechanics-compatible equations), is that it is the result of disciplined reasoning. Einstein created new physics.., but [his] work was done while satisfying certain severe restrictions ... a theory that satisfies such a broad constraints must, I think, be beautiful. (Nahin, 2006, p. xix).

Le Lionnais's ideas are more theoretically developed. He distinguishes between 'classical' and 'romantic' (Le Lionnais, 2004) beauty in mathematical propositions and methods. According to Le Lionnais a piece of mathematics possesses classical beauty "when we are impressed by its austerity or its mastery over diversity, and even more so when it combines these two characteristics in a harmoniously arranged structure" (Le Lionnais, 2004, p. 124). Regularity is the property more clearly associated with Le Lionnais' classical beauty. He thinks that the geometry of the triangle, cycloids and the logarithmic spiral exemplify classical beauty. In contrast, a piece of mathematics possesses romantic beauty when its beauty consists in the "glorification of violent emotion, non-conformism and eccentricity" (Le Lionnais, 2004, p. 130). The notion of asymptote, complex numbers, and Cantor's notion of infinity are some examples of romantic beauty. Le Lionnais's approach in terms of the art-historical distinction between classical and romantic[4] beauty is certainly interesting and more theoretically developed; but I believe it has important limitations: it deals with mathematical beauty not by offering an explication of aesthetic phenomena in mathematics, but by dealing with mathematics as an art; it resorts to the history of art rather than to an aesthetic theory. The distinction between classical and romantic art refers to a difference in style; it does not

[4] Western art movements are customarily classified by periods. Three of the most conspicuous pre-20th century periods are the Baroque, the Rococo or "Classical", and the Romantic. Le Lionnais adopts the standard distinction between classical and romantic movements in history of art, but the baroque is missing from his discussion. He does not give a reason for that, thus, Le Lionnais's approach seems to leave some room for us to ask about the existence of "baroque" mathematical beauty and how should it be characterized.

refer to the nature of the aesthetic phenomenon but to some differences among classes of objects that embody that phenomenon.

I think that the question of how to interpret what mathematicians mean when they use aesthetic expressions is best addressed in the context of an aesthetic theory that deals with more fundamental and general issues. Approaching mathematics as if it were an art seems to me a little like rushing matters. Perhaps it is better to start by addressing basic issues. This is precisely my intention: I shall develop the basis of an aesthetics of mathematics. To address this task it is a good idea to focus further on our problem and clarify the conceptual apparatus that will be employed to tackle it.

Let us identify our task: I shall address the problem of giving a literal interpretation of aesthetic judgements in mathematics. Furthermore, I shall address this problem from the point of view of contemporary mathematical practice. This means that the mathematical beauty that is our subject matter is not the beauty that the discipline as a whole possesses, as a result of, for example, mathematics being a rigorous or disciplined endeavour. Rather, I shall deal with the phenomena involved in mathematician's judgements, like 'Cantor's notion of infinity is beautiful' or 'proofs by exhaustion are cumbersome'. I accept mathematicians' judgements and assume that beauty, ugliness, elegance, etc. are properties of some pieces of mathematics. Some pieces are beautiful, but some others are ugly, elegant, aesthetically neutral, etc. Hence, in addition to beauty, things like ugliness or elegance are also part of the subject matter of this work.

Let us now clarify the conceptual apparatus that will be employed. Two elements are relevant here: the theoretical tenets that will be endorsed and the methodologies that will be employed. I consider that a healthy trend in contemporary philosophy and aesthetics is that they no longer distance themselves from natural science. My approach will thus cohere as much as possible with empirical findings. I aim to be consistent with science, and, when no scientific results are available, to employ a scientifically informed common sense. Philosophically, this work sympathizes with analytic philosophy's commitment to precision, thoroughness and conceptual rigour, and thus with methods like formalization and conceptual analysis. Despite this, due to the fact that mathematical aesthetic experience will be addressed here in a detailed way for the very first time (as far as I know), it remains difficult to achieve its theoretical development or to engage in rigorous analytic discussion. My approach to aesthetic experience is thus descriptive (after all, any theory or discussion must start somewhere).

Within this framework I shall tackle the task of providing a literal interpretation of aesthetic judgements in mathematics by proposing an aesthetic theory that allows us to interpret *mathematical* aesthetic judgement as bona fide aesthetic judgements. The aesthetic theory I propose interprets aesthetic events as elements of aesthetic processes. The working of such processes will be explicated by presenting accounts of three central topics: aesthetic experience, aesthetic value and aesthetic judgement.

Now, the ideas that help to develop this theory have been borrowed from several sources. Insights, concepts and theories of authors working in aesthetics, philosophy of mathematics and philosophy of science will be employed in various ways: Peter Kivy's account of musical appreciation and Jenefer Robinson's theories on emotion and expression have provided inspiration for my approach to aesthetic experience and the aesthetic as aesthetic-process. Gian-Carlo Rota's insights about mathematical beauty are incorporated as part of my interpretation of aesthetic mathematical intentional object. James McAllister's account of the evolution of aesthetic preferences has been borrowed to develop my account of the evolution of aesthetic value. The theoretical aesthetic approaches of Isabel Hungerland and Peter Kivy on aesthetic terms, Nelson Goodman on metaphor and Alan Goldman on aesthetic value are used to develop my theoretical accounts of aesthetic terms, descriptions, judgement and value in the context of an aesthetic-process. In particular, Kivy and Robinson's works provide a valuable perspective from the point of view of aesthetics and, more importantly, inspiration on how to address some issues (Kivy's formalism inspires my analysis of experience and Robinson's emotions-as-processes theory inspires my idea of aesthetic-process). Rota's, McAllister's and Hungerland's ideas are incorporated into my approach in a more direct way, as their accounts of the role of knowledge, the dynamics of preferences, and the role of aesthetic terms form the point of departure of my own accounts. Perhaps a broader picture of this work is in order; the following is a summary of the chapters.

Summary of Chapters:

Chapters 1 and 2 present background discussions and surveys: there we find valuable insights into topics like beauty in mathematics and science, musical formalism, emotions and expression in art. Chapter 3 is the core of the work, it contains my proposal for an aesthetic theory for mathematics; Chapter 4 tests the theory by applying it to concrete examples.

In Chapter 1 I examine theoretical approaches to beauty in mathematics and science. I begin by surveying the work of two early 18th century authors: Lord Shaftesbury (Shaftesbury, 1711) and Francis Hutchenson (Hutchenson, 1738). Shaftesbury addresses beauty in numbers using the idea that order is the principle of beauty. For Hutchenson beauty is an idea that is aroused in our minds by the property of uniformity amidst variety; this principle accounts for the beauty of theorems. The common problem with these accounts is that they lead one to conclude that every mathematical item is beautiful. However, mathematicians would agree that there is ugly mathematics; there are elegant as well as clumsy proofs, for example. Although illuminating, Shaftesbury's and Hutchenson's approaches do not really address the notion of mathematical beauty as it is employed by mathematicians. I thus turn to the ideas of the mathematician Gian-Carlo Rota (Rota, 1997), who claims that 'mathematical beauty' is a term mathematicians employ to refer to the enlightenment provided by some pieces of mathematics. I discuss some of the shortcomings of Rota's non-literal approach. For example, this approach cannot account for the fact that mathematicians (experts in employing and introducing exotic terms and meanings) choose to employ the term 'beauty' rather than a less confusing term. After listening to mathematicians I then turn to philosophy of science with a survey of James McAllister's ideas (McAllister, 1996). McAllister interprets scientific change in terms of aesthetic canons rather than in terms of Kuhnian paradigms. McAllister's most attractive insight is the idea of the aesthetic induction, which accounts for historical changes in aesthetic preferences: preferences for certain properties of scientific theories increase as a scientific community witnesses recurrent appearances of those properties in empirically adequate theories. Theo Kuipers (Kuipers, 2004) further substantiates aesthetic induction by interpreting it in terms of the *mere exposure effect*; which consists in the unconscious development of preferences for familiar stimuli rather than for unfamiliar ones. The problem with aesthetic induction in the work of both authors is that it renders a purely *a posteriori* view of aesthetic judgements. In such a picture all preferences seem to depend on past experiences. This approach neglects autonomic affective responses that do not depend on a process of 'learning' responses; in emotions, for example. There are some other theoretical shortcomings to aesthetic induction; I use the work of the aestheticians Isabel Hungerland (Hungerland, 1962) and Peter Kivy (Kivy, 1975) on the problem of aesthetic terms to highlight the limitations of aesthetic induction as an aesthetic theory. This makes it clear that a consistent notion of aesthetic judgements in mathematics is in order (a subject to be addressed in Chapter 3).

Chapter 2 surveys two central and complementary ideas in aesthetics: form and (the expression of) emotions. This chapter is less critical than Chapter 1; its main aim is to collect ideas and, especially, inspiration. The first part of the chapter is devoted to Peter Kivy's musical formalism; the second part to Jenefer Robinson's theories of emotions and expression.

As we have seen, mathematics and music have enjoyed an old and close relationship; philosophy of music is thus a good place to look for clues as to how to address aesthetics in the highly formalized discipline of mathematics. Peter Kivy's musical formalism[5] (Kivy, 2002) is a rich source of ideas. Musical formalism is the view that the aesthetic value of music is determined by its form. In Kivy's view, purely instrumental music consists in sound structures that can be seen as plots without content. He establishes an analogy between music and following a story; the active involvement of our intellect is central in the experience of music. The survey of Kivy's ideas provides insights that I employ to develop my own approach: first, the very idea that form, rather than emotive content, plays a central role in musical experience[6]. For Kivy, music is the object that occupies the attention of the listener and the formal properties of this object is what characterizes music. Second, in musical experience intellectual activity plays an important function. Some of our pleasure in listening to music comes from the intellectual activities performed in the grasping of sound structures. In addition to Kivy's

[5] Kivy himself believes that the term 'formalism' is ill-chosen, but he accepts it, so I do, too.

[6] I propose my own interpretation of formalism, better suited to my approach: the function of form is to unify our aesthetic experience by providing a focus for our attention and causing affective reactions.

ideas, Rogers Scruton's view on the ontology of music, that music occurs in a special kind of space, helps to inspire my treatment of mathematical items as intentional objects in a phenomenological space.

Formalism is an approach that reacts to the idea that the expression of emotions is the aim of art. In order to complete our picture of aesthetic experience, in the second part of Chapter 2 I will survey Jenefer Robinson's approach to expression of emotions in art. Jenefer Robinson's (Robinson, 2004) theories of emotion and expression employ ideas from philosophy, cognitive psychology, and neurophysiology, offering an attractive synthesis of philosophical and empirical results. Robinson conceives of emotions as processes which involve psychological, physiological and behavioural events; this theory of emotion enables her to propose a theory of expression in art, which is that expression elucidates and individuates emotions in an imaginary character, a *persona*[7]: an object expresses an emotion when it can be interpreted as being deliberately designed to appear as evidence of the presence of an emotion. The survey of Robinson's ideas has provided the following insights: her attention to empirical results in emotional phenomena provides some background to constrain (or, rather, to inform) my approach to aesthetic experience. Robinson's theories serve to ground a useful relation between cognition and aesthetics by acknowledging the role of judgements and knowledge in emotion and its expression. Robinson's approach to emotion as process integrates aspects of cognitive and non-cognitive approaches to emotion; I borrow this process-approach to develop my aesthetic-process approach, which attempts to integrate different aspects of the aesthetic in mathematics. Finally, the emphasis of Robinson's expression theory on elucidation and individuation inspires my account of aesthetic judgements as articulators of aesthetic experience.

The insights gained in chapters 1 and 2 lay the foundation for the development of my own account. Chapter 3 presents my proposal for an aesthetics of mathematics. Since mathematics is not a traditional subject of aesthetics, the best way of making up for this absence of a tradition is to offer a theoretical justification for the aesthetic character of mathematical aesthetic judgements, and to further substantiate the theory with an *a posteriori* justification by applying it to concrete cases. In Chapter 3 I propose the basics

[7] Robinson does not attribute emotions to any concrete individual but to this imaginary person, the persona.

of an aesthetic theory which allows us to understand mathematical aesthetic judgements as literal aesthetic judgements. In short, my proposal is to see the different kinds of aesthetic things – such as aesthetic experience, aesthetic value, aesthetic descriptions, etc. – as related by the fact that they all are elements of a process in which objective properties, subjective reactions and social influences and contexts interact with each other; I call this process an *aesthetic-process*. This idea allows me to interpret the notion of 'the aesthetic' as a predicate that is applied meaningfully to the kinds of things that characteristically participate in aesthetic-processes. Aesthetic events should not be understood in isolation but as part of a process, of a system that develops by following different paths over different times. Among the events involved in an aesthetic-process I interpret three central ones: aesthetic experience, aesthetic value and aesthetic judgements.

Aesthetic experience is interpreted as a process that involves changes in the focus and content of our attention and the eliciting of affective responses. Aesthetic experience, due to the very nature of experience, is a particularly difficult issue to address. Aesthetic judgements in mathematics can be identified by the public use of aesthetic terms, and aesthetic value can be inferred and (as we shall see in Chapter 3) 'tracked' by using public aesthetic judgement. But aesthetic experiences are private; this fact perhaps explains the absence of an analysis of it. As a result, my approach to aesthetic experience is almost completely descriptive rather than theoretical or analytical.

Aesthetic value is interpreted as a relation between sets of properties and subjective reactions; its evolution is governed by a mechanism similar to aesthetic induction (I call it constrained aesthetic induction). Aesthetic judgements are interpreted as expressions of subjective states, characterized by the application of aesthetic terms, the role of which is to articulate aesthetic experience and to share the result of such articulation.

This theory allows us to address the question 'what do mathematicians mean when they talk about beauty?' Mathematicians, as I anticipated, employ aesthetic terms in a literal sense, but their use is closer to the way aesthetic terms are used by specialists, critics or artists, rather than by a person in the street. Appreciation depends profoundly on a great deal of background mathematical knowledge. The theory allows us to reach our final aim of interpreting aesthetic judgements in mathematics as follows: *mathematical aesthetic judgements are articulated expressions of subjective states (aesthetic experiences) which result from an affective engagement of our attention to a mathematical item. The affective reaction reflects our preferences (our aesthetic values), which in turn are modulated by our natural tendencies and cultural influences (constrained aesthetic induction)*.

Chapter 4 examines three examples of mathematical beauty. These examples serve to show how the theory developed in Chapter 3 can be applied. The first example addresses mathematical beauty by means of a very basic example proposed by Le Lionnais: the function $y=e^x$. The second example addresses a more refined judgement. I analyze an *elegant* proof: Cantor's diagonal argument. This example allows us to see how my ideas of aesthetic experience and articulation can account for the nuances involved in the application of the closely related notions of beauty and elegance. Finally, I address mathematical ugliness; by doing so we will have covered all the most relevant aesthetic terms employed in mathematics. I discuss the case of the computer-assisted proof of the four colour theorem. This example also serves to display the advantages of my systemic approach, since we shall see that aesthetic judgements in mathematics depend not only on inductive changes in *value* but also on changes in the nature of our *experience*. I also show that my theory is able to make predictions: I conjecture that computer-assisted proofs have little chance of being regarded as beautiful in the future, as has been conjectured by James McAllister (McAllister, 2005 pp. 28-29).

I have also included five appendices which include complementary material that furthers the discussion of musical formalism and theories of emotion and expression in Chapter 2. Such appendices are not necessary to understand the main ideas of this work, but the curious reader can find interesting information in them. Appendices 2 and 5, in particular, are devoted to addressing the problem of providing a clearer notion of musical form, in a spirit similar to my idea of aesthetical mathematical intentional objects. Although these appendices are interesting as samples of earlier stages in the development of the ideas I endorse here, they are not necessary to understand them.